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NOTE

On the Calculation of Coupling Coefficients in Amplitude Equations

1. INTRODUCTION

Amplitude equations are often used to describe marginally unstable phenomena in physical systems. Allowing for spatial modulation, the canonical forms for these equations are the (in general, coupled) Ginzburg–Landau equations. The form of these equations is universal but the sets of coefficients vary. To treat a specific problem, it is necessary to calculate the values of the coefficients appearing in the amplitude equations, and it is desirable to have an efficient and reliable way to do this.

It has also been recently realized that many solutions observed in numerical simulations of partial differential equations can be explained by analyzing the dynamics in neighborhoods of multiple bifurcation points. This strategy has been advocated by Golubitsky and Guckenheimer [1], among others, and has been found useful in a number of examples. Such problems may involve defective eigenspaces, and there is a need for an efficient reduction procedure to obtain coupling coefficients in the amplitude equations in these cases as well.

In the context of fluid mechanics, Ginzburg–Landau equations were first derived by Stewartson and Stuart [2] and by Newell and Whitehead [3]. The method of their derivation also suggests the algorithm for calculation of parameters that appear in equations. The straightforward way of calculation of coupling coefficients in the case of a single amplitude equation consists of the following main steps:

(a) find the neutral eigenmode of the linear eigenvalue problem.

(b) solve the inhomogeneous linear equations corresponding to second harmonics.

(c) find the inhomogeneous term in the resonant equation.

(d) find the neutral eigenmode of the adjoint eigenvalue problem.

(e) find coupling coefficient from the sovability condition which requires that (c) be orthogonal to (d), a step which involves integration.

In this paper we describe a purely algebraic approach to the problem of finding coupling coefficients, which we have used and described previously, but without details, in Mahalov and Leibovich [4]. We show that under the assumption that the Q–R factorization or the Singular Value Decomposition (SVD) of matrices of linear problems is available, steps (d) and (e) can be replaced with one matrix-vector multiplication and one division of two complex numbers. There is no need for the explicit calculation of adjoint eigenfunctions of the linear problem. We note that the Q–R factorization is obtained for free after step (a) is completed, and once this step is accomplished, the computation of the coupling coefficients requires on the order of N^2 complex multiplications, where N is the number of the basis functions used in the discretization. Thus, our method provides substantial simplification of the standard computational procedure and reduces the number of operations. All steps of the method are purely algebraic, and numerical quadrature, which adds additional programming complexity and can degrade accuracy, is avoided.

2. DESCRIPTION OF THE ALGORITHM

We write the equations under investigation symbolically in the following form

$$\frac{d\mathbf{u}}{dt} = \mathbf{L}\mathbf{u} + \mathbf{N}(\mathbf{u}). \tag{1}$$

Here $\mathbf{L}$ is a linear partial differential operator on the spatial ($\mathbf{x}$) variables alone, and $\mathbf{N}$ is a nonlinear operator involving spatial differentiation. For definiteness, in this paper we consider equations with quadratic nonlinearity $\mathbf{N}(\mathbf{u})$, with the Navier–Stokes equations in mind. Without loss of generality, we assume that $\mathbf{u} = 0$ is the base state, the stability of which we wish to investigate. The linear problem

$$\mathbf{L}\mathbf{v} = \sigma\mathbf{v} \tag{2}$$

serves as the starting point for the weakly nonlinear calculations. Under the assumption (which is not essential for the method) that there are two homogeneous directions and one inhomogeneous direction (e.g., as in channel and pipe flows), we project Eq. (2) on the N vectors of the Galerkin basis in the inhomogeneous direction (typically, Chebyshev

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polynomials). Then we obtain an algebraic eigenvalue problem

$$\mathbf{A}(\alpha, \beta, R)\mathbf{w} = \sigma \mathbf{w}, \tag{3}$$

where α and β are wavenumbers in homogeneous directions, R is a (possibly vector-value) control parameter like the Reynolds number σ is a complex eigenvalue, $\mathbf{A}$ is a complex $N \times N$ complex matrix, and $\mathbf{w}$ is the eigenvector of Galerkin coefficients. The representation in the physical domain is given by $\mathbf{v} = \mathbf{T}^{-1}\mathbf{w}$, where $\mathbf{T}$ is the linear operator of the Chebyshev transformation and $\mathbf{T}^{-1}$ is the inverse Chebyshev transform.

On surfaces in the parameter space where the linear eigenproblem has simple complex conjugate roots with zero real part, there is a simple Hopf bifurcation. For the simple Hopf bifurcation the complex amplitude $a(t)$ satisfies the equation

$$\frac{da}{dt} = \sigma a + \lambda a\,|a|^2. \tag{4}$$

Here $a(t)$ is the amplitude of the weakly nonlinear disturbance, so $\mathbf{u} = a(t)\,\mathbf{v}(\mathbf{x}) +$ higher-order terms, and λ is a (complex) coupling coefficient known as the Landau constant. The latter is found from a solvability condition needed to eliminate resonant terms at third order in the expansions in powers of a and a^*. Allowing for spatial modulation, Eq. (4) becomes the Ginzburg–Landau equation

$$\frac{\partial a}{\partial \tau} = \sigma a + \mu \frac{\partial^2 a}{\partial \xi^2} + \lambda a\,|a|^2, \tag{5}$$

where τ is a slow time and ξ is a slow space variable. In the single Ginzburg–Landau equation there are four parameters that need to be computed: σ is found from (3), and μ also may be computed directly from the dispersion relation, by using, for example, a finite differences approximation. These coefficients are independent of the nonlinear term $\mathbf{N}(\mathbf{u})$. The calculation of λ in (5) is identical to the calculation of the coupling coefficient in the simple Hopf bifurcation equation (4). For a more detailed account of the theory of Ginzburg–Landau equations, one can consult Refs. [2] and [3].

Now we describe our algorithm. We expand $\mathbf{u}$ in Taylor series as follows:

$$\begin{aligned}\mathbf{u} &= a \exp(i(\alpha x + \beta y))\,\mathbf{v}_1(z)\\ &\quad + a^2 \exp(2i(\alpha x + uy))\,\mathbf{v}_2(z) + |a|^2\,\mathbf{v}_3(z)\\ &\quad + a\,|a|^2 \exp(i(\alpha x + \beta y))\,\mathbf{v}_4(z) + \cdots + \text{c.c.}\end{aligned}$$

Here $\mathbf{v}_1$ is the neutral eigenfunction computed at the critical point $R = R_c$, $\alpha = \alpha_c$, $\beta = \beta_c$; the $\mathbf{v}_j(z)$, $j > 1$ are unknown functions at this stage, and c.c. stands for complex conjugate. Galerkin coefficients of $\mathbf{v}_j$ are found using Chebyshev transforms. The vectors $\mathbf{w}_j = \mathbf{T}\mathbf{v}_j$ give us the representation of $\mathbf{v}_j$ in the spectral domain. The algorithm consists of solving a sequence of linear inhomogeneous algebraic equations corresponding to different powers of the amplitudes. The novelty of the method is the use of Q–R factorization or Singular Value Decomposition (SVD) in the calculation of coupling coefficients. This technique avoids the explicit need to compute adjoint eigenfunctions and the potentially inaccurate use of quadratures in the calculation of coupling coefficients. The inhomogeneous terms of linear equations can be generated very efficiently if the computer language incorporates symbolic computations.

Let us define Q–R factorization, with pivoting, of the matrices $\mathbf{B}_1 = (\mathbf{A}(\alpha, \beta, R) - i\omega\mathbf{I})$ and $\mathbf{B}_2 = (\mathbf{A}(2\alpha, 2\beta, R) - 2i\omega\mathbf{I})$ as follows:

$$\mathbf{B}_1 \Pi_1 = \mathbf{Q}_1 \mathbf{R}_1 \qquad \text{and} \qquad \mathbf{B}_2 \Pi_2 = \mathbf{Q}_2 \mathbf{R}_2.$$

Here $\mathbf{R}_1, \mathbf{R}_2$ are upper triangular matrices, $\mathbf{Q}_1, \mathbf{Q}_2$ are unitary matrices

$$\mathbf{Q}_j \mathbf{Q}_j^* = \mathbf{Q}_j^* \mathbf{Q}_j = \mathbf{I}\,(j = 1, 2),$$

and Π_1, Π_2 are permutation matrices. The coefficient of a^2 yields an equation of the following form

$$\{\mathbf{A}(2\alpha, 2\beta, R) - 2i\omega\mathbf{I}\}\,\mathbf{w}_2 = \mathbf{g}_2, \tag{6}$$

where the matrix $\mathbf{A}(2\alpha, 2\beta, R)$ comes from the linear stability problem and is evaluated for the critical values of parameters $R = R_c$, $\alpha = \alpha_c$, $B = B_c$. Under the assumption that there are no resonances at the second order, the matrix $\mathbf{A}(2\alpha_c, 2\beta_c, R_c) - 2i\omega\mathbf{I}$ is invertible. Then $\mathbf{w}_2$ can be found from the linear system of equations (6). The equation corresponding to $|a|^2$ is treated similarly.

Next we consider the coefficient of the term $a\,|a|^2$, which satisfies an equation of the type

$$[\mathbf{A}(\alpha_c, \beta_c, R_c) - i\omega\mathbf{I}]\,\mathbf{w}_4 = \lambda \mathbf{w}_1 + \mathbf{g}_1. \tag{7}$$

Note that the linear operator $[\mathbf{A}(\alpha_c, \beta_c, R_c) - i\omega\mathbf{I}]$ is not invertible and, therefore, this system of equations cannot be solved for an arbitrary right hand side, reflecting a resonant condition. The coefficient λ is found from the condition that Eq. (7) be solvable. In the representation $\mathbf{B}_1 \Pi_1 = \mathbf{Q}_1 \mathbf{R}_1$ we have $\mathbf{R}_1(N, N) = 0$ and $\mathbf{R}_1(j, j) \neq 0$ $(j = 1, 2, 3, \ldots, N-1)$, since the null space of $\mathbf{B}_1$ is one-dimensional. The coupling coefficient λ therefore can be immediately found as

$$\lambda = -\frac{(\mathbf{Q}_1^* \mathbf{g}_1)_N}{(\mathbf{Q}_1^* \mathbf{w}_1)_N},$$

where N is the number of basis functions used in the inhomogeneous direction and $\mathbf{Q}_1^*$ is the matrix adjoint to $\mathbf{Q}_1$.

Another way to compute coupling coefficients is to use the Singular Value Decomposition (SVD). The SVD is numerically very robust and we prefer this decomposition over Q–R factorization in cases with multiple or defective eigenvalues (e.g., the nonsemisimple double Hopf bifurcation considered in Mahalov and Leibovich [6] at which the eigenvalues are imaginary and have algebraic multiplicity two, and geometric multiplicity one). For a more detailed account of the theory of these decompositions and a comparison of their performance, one can consult Golub and Van Loan [7]. We define Singular Value Decompositions of the matrices of the linear problems

$$\mathbf{B}_1 \equiv \mathbf{A}(\alpha_c, \beta_c, R_c) - i\omega \mathbf{I} = \mathbf{S}_1 \mathbf{D}_1 \mathbf{Q}_1^*$$
$$\mathbf{B}_2 \equiv \mathbf{A}(2\alpha_c, 2\beta_c, R_c) - 2i\omega \mathbf{I} = \mathbf{S}_2 \mathbf{D}_2 \mathbf{Q}_2^*.$$

Here $\mathbf{D}_j$ are diagonal matrices of singular values and $\mathbf{S}_j$ and $\mathbf{Q}_j$ are unitary matrices. We have $\mathbf{D}_1(N, N) = 0$ and $\mathbf{D}_1(j, j) \neq 0$ $(j = 1, 2, 3, ..., N-1)$ since the null space of $\mathbf{B}_1$ is one-dimensional. An algorithm for the calculation of coupling coefficients using SVD is given in the Appendix, written in the Mathematica style. It is worth noticing that the structure of the program is extremely simple. Unless the nonlinear term is defined incorrectly, there are very few places where the programmer can make a mistake. The coupling coefficient λ is easily found from the resonant equation as

$$\lambda = -\frac{(\mathbf{S}_1^* \mathbf{g}_1)_N}{(\mathbf{S}_1^* \mathbf{w}_1)_N}, \tag{8}$$

where N is the number of basis functions and $\mathbf{S}_1^*$ is the matrix adjoint to $\mathbf{S}_1$.

Suppose now that $\mathbf{e}$ is the error in the calculation of $\mathbf{g}_1$. Then the error in the calculation of $\mathbf{S}_1^* \mathbf{g}_1$ is $\mathbf{S}_1^* \mathbf{e}$. Since $\mathbf{S}_1^*$ is the unitary matrix, we have $\|\mathbf{S}_1^* \mathbf{e}\| = \|\mathbf{e}\|$, where $\|\cdot\|$ is the Euclidean norm. Thus, the matrix-vector multiplication in (8) does not amplify the error.

We note that the Chebyshev transformation in the algorithm described in the Appendix can be accomplished in $N \log N$ complex multiplications. As a consequence, the most time consuming operations in the method are the matrix-vector multiplications. These multiplications require of the order of N^2 complex multiplications with N basis functions used in the discretization. Thus, the total number of complex multiplications in the algorithm is of the order N^2.

TABLE I

Ph(λ)—Absolute Value of the Phase of the Coupling Coefficient λ

N:	30	35	40	45	50
Ph(λ):	2.49479788	2.49479616	2.49479610	2.49479609	2.49479608

Note. $R = 1066$, Reynolds number; $\Omega = 26.96$, rotation rate; $\alpha = 0.1$ – axial wavenumber; $\beta = 1$ – azimuthal wavenumber. N is the number of Chebyshev polynomials used in the inhomogeneous direction.

TABLE II

Ph(λ)—Absolute Value of the Phase of the Coupling Coefficient λ

N:	30	35	40	45	50
Ph(λ):	3.11880686	3.11880684	3.11880682	3.11880682	3.11880682

Note. $R = 83.1$, Reynolds number; $\Omega = 415$, rotation rate; $\alpha = -0.1$ – axial wavenumber; $\beta = 1$ – azimuthal wavenumber. N is the number of Chebyshev polynomials used in the inhomogeneous direction.

3. ACCURATE CALCULATION OF COUPLING COEFFICIENTS IN ROTATING PIPE FLOWS AND IN PLANE POISEUILLE FLOW

In this section we report accurate calculation of coupling coefficients for rotating pipe flow and for plane Poiseuille (channel) flow. We note that the value of the coupling coefficient depends on the normalization of the neutral eigenmode, but the phase of complex number λ is an invariant characteristic (up to a sign).

In the case of the rotating pipe flow, the basic laminar flow has a parabolic velocity profile with maximum speed W_0 on the center-line; r_0 is the pipe radius and Ω_0 its angular velocity. With r_0 and r_0^2/ν as units of distance and time, the problem depends on the rotational Reynolds number, Ω, and an axial Reynolds number, R, defined by $\Omega = \Omega_0 r_0^2/\nu$, $R = r_0 W_0/\nu$. The coupling coefficient λ was computed for two values of parameters R and Ω using the numerical algorithm described above. The first of the choices for R and Ω for these particular calculations, $R = 1066$ and $\Omega = 26.96$, corresponds to the minimum rotation rate for neutral stability. The second, $R = 83.1$ and $\Omega = 415$ corresponds to the minimum Reynolds number for neutral stability. In Tables I and II we present results of our calculations with $N = 30$, 35, 40, 45, and 50 Chebyshev functions used in the inhomogeneous direction for each component of the velocity and pressure fields.

In the case of the channel flow with parabolic velocity profile we define the Reynolds number R based on the maximum velocity of the base flow and the channel halfwidth. The problem may be formulated in terms of a streamfunction. The critical Reynolds number is 5772.22 and the critical axial wavenumbers are $\alpha = 1.02056$ and $\beta = 0$. The coupling coefficient for this point on the neutral stability

TABLE III

Ph(λ)—Absolute Value of the Phase of the Coupling Coefficient λ

N:	40	45	50	55	60
Ph(λ):	1.39045107	1.39158442	1.39157415	1.39157118	1.39157113

Note. $R = 5772.22$, Reynolds number; $\alpha = 1.02056$ – axial wavenumber; $\beta = 0$ – axial wavenumber. N is the number of Chebyshev polynomials used in the inhomogeneous direction.

curve was computed with $N=40$, 45, 50, 55, and 60 Chebyshev functions used to represent the streamfunction in the inhomogeneous direction. The data is presented in Table III.

4. CONCLUSIONS

The extension of the algorithm to the case of coupled amplitude equations (including coupled Ginzburg–Landau equations) is straightforward, and an example is given in [6]. For example, for two interacting amplitudes a and b the function $\mathbf{u}$ is expanded in the Taylor series with terms of the form $a^j(a^*)^k b^m(b^*)^n$. After that, one needs to solve a sequence of linear inhomogeneous equations corresponding to different powers of the amplitudes. The coupling coefficients are found with the method described above from the resonant equations using Q–R factorization or SVD of matrices of the linear stability problem.

APPENDIX

```
(*                                                      *)
(* Read data corresponding to the                       *)
(* unitary matrices S_j and Q_j.                        *)
(*                                                      *)
S0 = << S0.data; S1 = << S1.data; S2 = << S2.data;
Q0 = << Q0.data; Q1 = << Q1.data; Q2 = << Q2.data;
(* Read singular values D_j.                            *)
D0 = << D0.data; D1 = << D1.data; D2 = << D2.data;
(*                                                      *)
(* Define the nonlinear term.                           *)
(*                                                      *)
N[u_] := nonlinear term of the equation (1);
(*                                                      *)
(* Find the representation in the physical domain       *)
(* of the inhomogeneous term in the                     *)
(* equation for a^2.                                    *)
(*                                                      *)
f2 = Coefficient[N[u], a^2];
(*                                                      *)
(* Find the representation in the spectral domain       *)
(* of the inhomogeneous term in the equation            *)
(* for a^2 using Fast Chebyshev Transform.              *)
(*                                                      *)
g2 = T[f2];
(*                                                      *)
(* Find the representation in the spectral domain       *)
(* of the second harmonic corresponding to a^2.         *)
(* This step requires the following operations:        *)
(* matrix-vector multiplication with the adjoint of S2, *)
(* division of each component of the resulting          *)
(* vector by the corresponding singular value,          *)
(* matrix-vector multiplication with the matrix Q2.     *)
(*                                                      *)
w2 = Q2((S2* g2)/D2);
(*                                                      *)
(* Find the representation in the physical domain       *)
(* of the second harmonic corresponding to              *)
(* a^2 using Inverse Fast Chebyshev Transform.          *)
(*                                                      *)
v2 = T^-1[w2];
(*                                                      *)
(* Repeat previous steps for the equation               *)
(* corresponding to |a|^2.                              *)
(*                                                      *)
(* Find the representation in the physical domain       *)
(* of the inhomogeneous term in the equation for a |a|^2. *)
(*                                                      *)
f1 = Coefficient[N[u], a |a|^2];
(*                                                      *)
(* Find the representation in the spectral domain       *)
(* of the inhomogeneous term in the equation            *)
(* for a |a|^2 using Fast Chebyshev Transform.          *)
(*                                                      *)
g1 = T[f1];
(*                                                      *)
(* Find the coupling coefficient λ.                     *)
(*                                                      *)
λ = -((S1*) g1)[N]/((S1*) w1)[N];
```

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